The Energy-momentum of a Poisson structure

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Abstract

Consider the quasi-commutative approximation to a noncommutative geometry. It is shown that there is a natural map from the resulting Poisson structure to the Riemann curvature of a metric. This map is applied to the study of high-frequency gravitational radiation. In classical gravity in the WKB approximation there are two results of interest, a dispersion relation and a conservation law. Both of these results can be extended to the noncommutative case, with the difference that they result from a cocycle condition on the high-frequency contribution to the Poisson structure, not from the field equations.

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1 Introduction and motivation

Our purpose here is to show that a gravitational field is intimately associated with a lack of commutativity of the local coordinates of the space-time structure on which the field is defined. The gravitational field can be described by a moving frame with local components e^μ_α; the lack of commutativity by a commutator $J^{\mu\nu}$. To a coordinate x^μ one associates a conjugate momentum p_α and to this couple a commutator $[p_\alpha, x^\mu]$. The relation we are seeking then can be succinctly written as the identity

$$[p_\alpha, x^\mu] = e^\mu_\alpha. \tag{1.1}$$

After some preliminary mathematics we shall be in a position to state the relation in more detail.

Let μ be a typical 'large' source mass with 'Schwarzschild radius' $G_N\mu$. We have two length scales, determined by respectively the square $G_N\hbar$ of the Planck length and by $\bar{k}$. The gravitational field is weak if the dimensionless parameter $\epsilon_{GF} = G_N\hbar\mu^2$ is small; the space-time is almost commutative if the dimensionless parameter $\epsilon = \bar{k}\mu^2$ is small. These two parameters are not necessarily related but we shall here assume that they are of the same order of magnitude,

$$\epsilon_{GF} \simeq \epsilon. \tag{1.2}$$

If noncommutativity is not directly related to gravity then it makes sense to speak of ordinary gravity as the limit $\bar{k} \to 0$ with $G_N\mu$ nonvanishing. On the other hand if noncommutativity and gravity are directly related then both should vanish with $\bar{k}$. We wish here to consider an expansion in the parameter ϵ, which we have seen is a measure of the relative dimension of a typical 'space-time cell' compared with the Planck length of a typical quantity of gravitational energy. Our motivation for considering noncommutative geometry as an 'avatar' of gravity is the belief that it sheds light on the role [1] of the gravitational field as the universal regulator of ultra-violet divergences. We give a brief review of the approach we use to study gravitational fields on Lorentz-signature manifolds but only in so far as it is necessary. A general description can be found elsewhere [2] as can a simple explicit solution [3].

We introduce a set $J^{\mu\nu}$ of elements of an associative algebra $\mathcal{A}$ ('noncommutative space' or 'fuzzy space') defined by commutation relations

$$[x^\mu, x^\nu] = i\bar{k}J^{\mu\nu}(x^\sigma). \tag{1.3}$$

The constant $\bar{k}$ is a square of a real number which defines the length scale on which the effects of noncommutativity become important. The $J^{\mu\nu}$ are restricted by Jacobi identities; we shall see below that there are two other requirements which also restrict them.

We suppose the differential calculus over $\mathcal{A}$ to be defined by a frame, a set of 1-forms θ^α which commute with the elements of the algebra. We assume the derivations dual to these forms to be inner, given by momenta p_α as in ordinary quantum mechanics

$$e_\alpha = \operatorname{ad} p_\alpha. \tag{1.4}$$

Recall that here the momenta p_α include a factor $(i\bar{k})^{-1}$. The momenta stand in duality to the position operators by the relation (1.1). However, now consistency relations in the algebra restrict θ^α and $J^{\mu\nu}$. Most important thereof is the Leibniz rule which defines differential relations

$$i\bar{k}[p_\alpha, J^{\mu\nu}] = [x^{[\mu}, [p_\alpha, x^{\nu]}]] = [x^{[\mu}, e^{\nu]}_\alpha] \tag{1.5}$$

between the $J^{\mu\nu}$ on the left and the frame components e^μ_α on the right.

Now we can state the relation (1.1) between noncommutativity and gravity more precisely. The right-hand side of this identity defines the gravitational field. The left-hand side must obey Jacobi identities. These identities yield relations between quantum mechanics in the given curved space-time and the noncommutative structure of the algebra. The three aspects of reality then, the curvature of space-time, quantum mechanics and the noncommutative structure are intimately connected. We shall consider here an even more exotic possibility that the field equations of general relativity are encoded also in the structure of the algebra so that the relation between general relativity and quantum mechanics can be understood by the relation which each of these theories has with noncommutative geometry.

We resume the various possibilities in a diagram, starting with a classical metric $\tilde{g}_{\mu\nu}$.

$$\begin{array}{ccccc}
\tilde{g}_{\mu\nu} & \longrightarrow & \tilde{\theta}^\alpha & \longleftrightarrow & \tilde{\Lambda}^\alpha_\mu \\
 & & \downarrow & & \\
g_{\mu\nu} & \longleftarrow & \theta^\alpha & \longleftrightarrow & \Omega(\mathcal{A}) \\
 & & \downarrow & & \\
 & & J^{\mu\nu} & \longrightarrow & \mathcal{A}
\end{array} \tag{1.6}$$

The most important flow of information is from the classical metric $\tilde{g}_{\mu\nu}$ to the commutator $J^{\mu\nu}$, defined in three steps. The first step is to associate to the metric a moving frame $\tilde{\theta}^\alpha$, which can be written in the form $\tilde{\theta}^\alpha = \tilde{\theta}^\alpha_\mu dx^\mu$. The frame is then 'quantized' according to the ordinary rules of quantum mechanics; the dual derivations $\tilde{e}_\alpha$ are replaced by inner derivations $e_\alpha = \mathrm{ad}\, p_\alpha$ of a noncommutative algebra. The commutation relations are defined by the $J^{\mu\nu}$, obtained from the θ^α by solving a differential equation. If the space is flat and the frame is the canonical flat frame then the right-hand side of (1.5) vanishes and it is possible to consistently choose $J^{\mu\nu}$ to be constant; the map (1.7) is not single valued since any constant $J^{\mu\nu}$ has flat space as inverse image. If the noncommutative structure is defined by a twist S then the latter can be used to define a twisted derivation of the algebra. In many cases it can be shown to be equivalent to a frame derivation which satisfies the ordinary Leibniz rule.

There are in fact three enmeshed problems to consider. The first is the map

$$J^{\mu\nu} \longrightarrow \mathrm{Curv}(\theta^\alpha) \tag{1.7}$$

from the Poisson structure $J^{\mu\nu}$ to the curvature of a frame. It allows us to express the Einstein tensor in terms of $J^{\mu\nu}$. The interest at the moment of this point is limited by the fact that we have no 'equations of motion' for $J^{\mu\nu}$.

The second problem is the mode decomposition of the image metric. We shall see that in the linear approximation there are three modes in all, which we shall consider as the two dynamical modes of a spin-2 particle plus a scalar mode. They need however not all be present: the graviton will be polarized by certain background noncommutative 'lattice' structure. This leads to the problem of the propagation of the modes in the 'lattice'. The covariant WKB approximation [4, 5, 6] applied to the Einstein equations is an elegant method to study the propagation of gravitational waves in a given background. We shall mimic it here as far as possible.

The third problem we wish to aboard here is that of the existence and definition of an energy-momentum for the Poisson structure and of an eventual contribution of this energy-momentum to the gravitational field equations. In the formulation which we are considering the Einstein tensor is determined by integrability conditions for the underlying associative-algebra structure, which suggests the possibility of interpreting the commutative limit of this tensor as the energy-momentum of the symplectic structure. We have no action for the metric; the field equations, we claim, are integrability conditions for the differential calculus. One of these conditions is the cocycle condition (2.30) given below, which is similar in structure to the condition that the Ricci tensor vanish; the two are however not equivalent.

This article is a natural sequel of a previous one [7]. The basic idea was also partially anticipated in a more specialized treatment [8] of asymptotically-flat space-times as well as in a string-theoretical reduction [9, 10] of noncommutative geometry to a supplementary 2-form, the Kalb-Ramond B-field, which appears within the context of super-string theory. There is also a definite overlap with an interesting recent interpretation [11] of the map (1.7) as a redefinition of the gravitational field in terms of noncommutative electromagnetism. We shall return to this interpretation in Section 2.6.

The paper is organized as follows. In Section 2 we derive the map (1.7). In Section 3 we discuss the mode decomposition using the WKB formalism. In Section 4 finally we propose a definition of the energy-momentum of the limiting Poisson structure. We use letters λ, μ, ν, ρ to denote the coordinate indices and $\alpha, \beta, \gamma, \zeta$ for the frame indices. In the example we use the frame metric $(-1, 1, 1, 1)$.

2 The Correspondence

To fix the notation we give briefly some elements of the noncommutative frame formalism. We refer to the literature [2, 7] for further details.

2.1 Preliminary formalism

We start with a noncommutative $*$-algebra $\mathcal{A}$ generated by four hermitian elements x^μ which satisfy the commutation relations (1.3). Assume that over $\mathcal{A}$ is a differential calculus which is such [2] that the module of 1-forms is free and possesses a preferred frame θ^α which commutes,

$$[x^\mu, \theta^\alpha] = 0 \tag{2.1}$$

with the algebra. The space one obtains in the commutative limit is therefore parallelizable with a global moving frame $\tilde{\theta}^\alpha$ defined to be the commutative limit of θ^α. We can write the differential

$$dx^\mu = e^\mu_\alpha \theta^\alpha, \qquad e^\mu_\alpha = e_\alpha x^\mu. \tag{2.2}$$

The differential calculus is defined as the largest one consistent with the module structure of the 1-forms so constructed. The algebra is defined by a product which is restricted by the matrix of elements $J^{\mu\nu}$; the metric is defined by the matrix of elements e^μ_α. Consistency requirements, essentially determined by Leibniz rules, impose relations between these two matrices which in simple situations allow us to find a one-to-one correspondence between the structure of the algebra and the metric. The input of which we shall make the most use is the Leibniz rule (1.5) which can also be written as relation between 1-forms

$$i\hbar dJ^{\mu\nu} = [dx^\mu, x^\nu] + [x^\mu, dx^\nu]. \tag{2.3}$$

One can see here a differential equation for $J^{\mu\nu}$ in terms of e^μ_α. In important special cases the equation reduces to a simple differential equation of one variable.

In addition, we must insure that the differential is well defined. A necessary condition is that $d[x^\mu, \theta^\alpha] = 0$, from which it follows that the momenta p_α must satisfy quadratic relation [2]. On the other hand, from (2.1) it follows that

$$d[x^\mu, \theta^\alpha] = [dx^\mu, \theta^\alpha] + [x^\mu, d\theta^\alpha] = e^\mu_\beta[\theta^\beta, \theta^\alpha] - \tfrac{1}{2}[x^\mu, C^\alpha{}_{\beta\gamma}]\theta^\beta\theta^\gamma, \tag{2.4}$$

where we have introduced the Ricci rotation coefficients

$$d\theta^\alpha = -\frac{1}{2} C^\alpha{}_{\beta\gamma}\theta^\beta\theta^\gamma. \tag{2.5}$$

Therefore we find that multiplication of 1-forms must satisfy

$$[\theta^\alpha, \theta^\beta] = \tfrac{1}{2}\theta^\beta_\mu [x^\mu, C^\alpha{}_{\gamma\delta}]\theta^\gamma\theta^\delta. \tag{2.6}$$

Using the consistency conditions we obtain that

$$\theta^{[\beta}_\mu [x^\mu, C^{\alpha]}{}_{\gamma\delta}] = 0, \tag{2.7}$$

and also that the expression $\theta^{(\alpha}_\mu [x^\mu, C^{\beta)}{}_{\gamma\delta}]$ must be central.

The metric is defined by the map

$$g(\theta^\alpha \otimes \theta^\beta) = g^{\alpha\beta}. \tag{2.8}$$

The bilinearity of the metric implies that $g^{\alpha\beta}$ are complex numbers. We choose the frame to be orthonormal in the commutative limit; we can write therefore

$$g^{\alpha\beta} = \eta^{\alpha\beta} - i\epsilon h^{\alpha\beta}. \tag{2.9}$$

We introduce also

$$g^{\mu\nu} = g(dx^\mu \otimes dx^\nu) = e^\mu_\alpha e^\nu_\beta g^{\alpha\beta}. \tag{2.10}$$

We write $g^{\mu\nu}$ as a sum

$$g^{\mu\nu} = g^{\mu\nu}_+ + g^{\mu\nu}_- \tag{2.11}$$

of symmetric and antisymmetric parts. To lowest order in the noncommutativity in general we have $h^{\alpha\beta} = -h^{\beta\alpha}$ so we find that

$$g^{\mu\nu}_+ = \tfrac{1}{2}\eta^{\alpha\beta}[e^\mu_\alpha, e^\nu_\beta]_+ - \tfrac{1}{2}i\epsilon h^{\alpha\beta}[e^\mu_\alpha, e^\nu_\beta] \tag{2.12}$$

and

$$g^{\mu\nu}_- = \tfrac{1}{2}\eta^{\alpha\beta}[e^\mu_\alpha, e^\nu_\beta] - \tfrac{1}{2}i\epsilon h^{\alpha\beta}[e^\mu_\alpha, e^\nu_\beta]_+. \tag{2.13}$$

We shall restrict our considerations in Section 2.3 to first-order perturbations of flat space. We set

$$g^{\mu\nu} = \eta^{\mu\nu} - \epsilon g^{\mu\nu}_1, \qquad e^\mu_\alpha = \delta^\mu_\alpha + \epsilon\Lambda^\mu_\alpha. \tag{2.14}$$

We have then the relations

$$g^{\mu\nu}_1 = -\eta^{\alpha\beta}\Lambda^{(\mu}_\alpha\delta^{\nu)}_\beta = -\Lambda^{(\mu\nu)}. \tag{2.15}$$

2.2 The quasi-commutative approximation

To lowest order in ϵ the partial derivatives are well defined and the approximation, which we shall refer to as the quasi-commutative,

$$[x^\lambda, f] = i\hbar J^{\lambda\sigma}\partial_\sigma f \tag{2.16}$$

is valid. The Leibniz rule and the Jacobi identity can be written in this approximation as

$$e_\alpha J^{\mu\nu} = \partial_\sigma e^{[\mu}_\alpha J^{\sigma\nu]}, \tag{2.17}$$

$$\epsilon_{\kappa\lambda\mu\nu}J^{\gamma\lambda}e_\gamma J^{\mu\nu} = 0. \tag{2.18}$$

We shall refer to these equations including their integrability conditions as the Jacobi equations.

Written in frame components the Jacobi equations become

$$e_\gamma J^{\alpha\beta} - C^{[\alpha}{}_{\gamma\delta}J^{\beta]\delta} = 0, \tag{2.19}$$

$$\epsilon_{\alpha\beta\gamma\delta}J^{\gamma\eta}(e_\eta J^{\alpha\beta} + C^\alpha{}_{\eta\zeta}J^{\beta\zeta}) = 0. \tag{2.20}$$

We have used here the expression for the rotation coefficients, valid also in the quasi-commutative approximation:

$$C^\alpha{}_{\beta\gamma} = \theta^\alpha_\mu e_{[\beta} e^\mu_{\gamma]} = -e^\nu_\beta e^\mu_\gamma \partial_{[\nu} \theta^\alpha_{\mu]}. \tag{2.21}$$

Inserting (2.19) into (2.20) one finds the relation

$$\epsilon_{\alpha\beta\gamma\delta} J^{\alpha\zeta} J^{\beta\eta} C^\gamma{}_{\eta\zeta} = 0. \tag{2.22}$$

One can solve (2.19) for the rotation coefficients. One obtains

$$J^{\gamma\eta} e_\eta J^{\alpha\beta} = J^{\alpha\eta} J^{\beta\zeta} C^\gamma{}_{\eta\zeta}, \tag{2.23}$$

or, provided J^{-1} exists,

$$C^\alpha{}_{\beta\gamma} = J^{\alpha\eta} e_\eta J^{-1}_{\beta\gamma}. \tag{2.24}$$

This can be rewritten as

$$C^\alpha{}_{\beta\gamma} = J^{\alpha\delta} e_\delta J^{\zeta\eta} J^{-1}_{\zeta\beta} J^{-1}_{\eta\gamma}. \tag{2.25}$$

From general considerations also follows that the rotation coefficients must satisfy the gauge condition

$$e_\alpha C^\alpha{}_{\beta\gamma} = 0. \tag{2.26}$$

Anticipating a notation from Section 2.3 we introduce

$$\hat{C}_{\alpha\beta\gamma} = J^{-1}_{\alpha\delta} C^\delta{}_{\beta\gamma}. \tag{2.27}$$

We find then that

$$\hat{C}_{\alpha\beta\gamma} = e_\alpha J^{-1}_{\beta\gamma} \tag{2.28}$$

and that

$$\hat{C}_{\alpha\beta\gamma} + \hat{C}_{\beta\gamma\alpha} + \hat{C}_{\gamma\alpha\beta} = 0, \tag{2.29}$$

an equation which is in fact obviously the same as (2.22) and which we can write as a de Rham cocyle condition

$$dJ^{-1} = 0, \qquad J^{-1} = \frac{1}{2} J^{-1}_{\alpha\beta} \theta^\alpha \theta^\beta. \tag{2.30}$$

One can write Equation (2.24) as

$$e_\alpha J^{-1}_{\beta\gamma} = J^{-1}_{\alpha\delta} C^\delta{}_{\beta\gamma}. \tag{2.31}$$

It follows that in the quasi-classical approximation the linear connection and therefore the curvature can be directly expressed in terms of the commutation relations. This is the content of the map (1.7). Using the expression

$$\omega_{\alpha\beta\gamma} = \tfrac{1}{2}(C_{\alpha\beta\gamma} - C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta}) \tag{2.32}$$

for the Ricci curvature tensor for example we obtain

$$
\begin{aligned}
2R_{\beta\zeta} = {} & J_{(\zeta\delta} e^\alpha e^\delta J^{-1}_{\beta)\alpha} + J^{\alpha\delta} e_{(\zeta} e_\delta J^{-1}_{\alpha\beta)} \\
& - J_{(\zeta}{}^\eta e^\alpha J^{-1}_{\eta\gamma} J^{\gamma\delta} e_\delta J^{-1}_{\beta)\alpha} + J^{\alpha\delta} e_\delta J^{-1}_{\eta\beta} J^{\eta\gamma} e_\gamma J^{-1}_{\alpha\zeta} \\
& + J_{\eta\delta} e^\delta J^{-1}_{\beta\alpha} J^{\eta\gamma} e_\gamma J^{-1\alpha}_{\zeta} + J^{\alpha\eta} e_{(\zeta} J^{-1}_{\eta\gamma} J^{\gamma\delta} e_\delta J^{-1}_{\beta)\alpha} \\
& - \tfrac{1}{2} J_{\zeta\delta} e^\delta J^{-1\alpha\eta} J_{\beta\gamma} e^\gamma J^{-1}_{\alpha\eta} + J^{\alpha\delta} e_\delta J^{-1}_{\alpha\eta} J_{(\zeta\gamma} e^\gamma J^{-1}_{\beta)}{}^{\eta}.
\end{aligned} \tag{2.33}
$$

We have here neglected the ordering on the right-hand side as it gives the corrections of second-order in ϵ. To understand better the relation between the commutator and the curvature in the following section we shall consider a linearization about a fixed 'ground state'.

2.3 The weak-field approximation

Assume then that we have a 'ground state' consisting of a J_0 and a corresponding image θ_0^α of the map

$$J_0 \to \mathrm{Curv}(\theta_0^\alpha) \tag{2.34}$$

which we can extend to a region around J_0. We assumed in Section 2.2 that the non-commutativity is small and we derived some relations to first-order in the parameter ϵ. We shall now make an analogous assumption concerning the gravitational field; we shall assume that ϵ_{GF} is also small and of the same order of magnitude. With these two assumptions the Jacobi equations become relatively easy to solve.

We suppose that the basic unknowns, J_0 and θ_0^α are constants and that they are perturbed to:

$$J^{\alpha\beta} = J_0^{\alpha\beta} + \epsilon I^{\alpha\beta}, \qquad \theta^\alpha = \theta_0^\beta(\delta^\alpha_\beta - \epsilon\Lambda^\alpha_\beta). \tag{2.35}$$

The leading order of the Jacobi system is given by

$$e_\gamma I^{\alpha\beta} - e_{[\gamma}\Lambda^{[\alpha}_{\delta]} J_0^{\beta]\delta} = 0, \tag{2.36}$$

$$\epsilon_{\alpha\beta\gamma\delta} J_0^{\gamma\eta} e_\eta I^{\alpha\beta} = 0. \tag{2.37}$$

Introducing the notation

$$\hat I_{\alpha\beta} = J_0^{-1}{}_{\alpha\gamma} J_0^{-1}{}_{\beta\delta} I^{\gamma\delta}, \qquad \hat\Lambda_{\alpha\beta} = J_0^{-1}{}_{\alpha\gamma}\Lambda^\gamma_\beta, \tag{2.38}$$

(2.36) can be written as

$$e_\gamma(\hat I_{\alpha\beta} - \hat\Lambda_{[\alpha\beta]}) = e_{[\alpha}\hat\Lambda_{\beta]\gamma} \tag{2.39}$$

and (2.37) as

$$\epsilon^{\alpha\beta\gamma\delta} e_\alpha \hat I_{\beta\gamma} = 0. \tag{2.40}$$

We note that $\hat I$ is a linear perturbation of J_0^{-1},

$$J^{-1}_{\alpha\beta} = J^{-1}_{0\alpha\beta} + \epsilon\hat I_{\alpha\beta}. \tag{2.41}$$

Equations (2.39-2.40) are the origin of the particularities of our construction, along with the fact that the 'ground-state' value $J_0^{\mu\nu}$ is an invertible matrix.

We decompose $\hat\Lambda$ as the sum

$$\hat\Lambda_{\alpha\beta} = \hat\Lambda^+_{\alpha\beta} + \hat\Lambda^-_{\alpha\beta} \tag{2.42}$$

of a symmetric and antisymmetric term. Constraint (2.39) can be written then as

$$e_\gamma \hat I_{\alpha\beta} - (e_\alpha \hat\Lambda^-_{\beta\gamma} + e_\beta \hat\Lambda^-_{\gamma\alpha} + e_\gamma \hat\Lambda^-_{\alpha\beta}) = e_\gamma \hat\Lambda^-_{\alpha\beta} + e_{[\alpha}\hat\Lambda^+_{\beta]\gamma}. \tag{2.43}$$

It implies a second cocyle condition. If we multiply by $\epsilon^{\alpha\beta\gamma\delta}$ we find that

$$\epsilon^{\alpha\beta\gamma\delta} e_\gamma \hat\Lambda^-_{\alpha\beta} = 0, \tag{2.44}$$

and Equation (2.43) simplifies to

$$e_\gamma(\hat I_{\alpha\beta} - \hat\Lambda^-_{\alpha\beta}) = e_{[\alpha}\hat\Lambda^+_{\beta]\gamma}. \tag{2.45}$$

This equation has the integrability conditions

$$e_\delta e_{[\alpha}\hat\Lambda^+_{\beta]\gamma} - e_\gamma e_{[\alpha}\hat\Lambda^+_{\beta]\delta} = 0. \tag{2.46}$$

But the left-hand side is the linearized approximation to the curvature of a (fictitious) metric with components $g_{\mu\nu} + \epsilon\hat\Lambda^+_{\mu\nu}$. If it vanishes then the perturbation is a derivative. (We have in fact shown here that a deformation of a commutator can be always chosen antisymmetric to first order). For some 1-form A with frame components A_α

$$\hat\Lambda^+_{\beta\gamma} = \tfrac{1}{2} e_{(\beta} A_{\gamma)}. \tag{2.47}$$

Equation (2.45) becomes therefore

$$e_\alpha(\hat I - \hat\Lambda^- - dA)_{\beta\gamma} = 0. \tag{2.48}$$

It follows then that for some 2-form c with constant components $c_{\beta\gamma}$

$$\hat\Lambda^- = \hat I - dA + c, \qquad \hat\Lambda_{\alpha\beta} = \hat I_{\alpha\beta} + e_\beta A_\alpha + c_{\alpha\beta}. \tag{2.49}$$

The remaining constraints are satisfied identically.

We can also introduce 2-forms

$$\hat I = \tfrac{1}{2}\hat I_{\alpha\beta}\theta^\alpha\theta^\beta, \qquad \hat\Lambda^- = \tfrac{1}{2}\hat\Lambda^-_{\alpha\beta}\theta^\alpha\theta^\beta \tag{2.50}$$

and write

$$d\hat I = 0, \qquad d\hat\Lambda^- = 0. \tag{2.51}$$

The first equation is a particular case of Equation (2.30). Because of the cocycle condition there must exist a 1-form C such that

$$\hat I_{\gamma\delta} = e_{[\gamma} C_{\delta]}. \tag{2.52}$$

We shall see below that the form of C depends partially on the choice of coordinates.

2.4 The map

We can now be more precise about the map (1.7) in the linear approximation. We recall that the fluctuations are redundantly parameterized by Λ^α_β. We can rewrite the map (1.7) as a map

$$\hat{I}^{\alpha\beta} \longrightarrow \Lambda^\alpha_\beta \tag{2.53}$$

from $\hat{I}^{\alpha\beta}$ to Λ^α_β. With $c_{\alpha\beta} = 0$ we have

$$\Lambda^\alpha_\beta = J_0^{\alpha\gamma}(\hat{I}_{\gamma\beta} + e_\beta A_\gamma). \tag{2.54}$$

As a perturbation of the frame $e^\mu_\alpha = \delta^\mu_\beta + \epsilon\Lambda^\mu_\alpha$ engenders a perturbation of the metric

$$g^{\mu\nu} = \eta^{\mu\nu} - \epsilon {g_1}^{\mu\nu}, \qquad {g_1}^{\mu\nu} = -\Lambda^{(\mu\nu)}, \tag{2.55}$$

it follows from (2.54) that

$$g_{1\alpha\beta} = -{J_{0(\alpha}}^\gamma(\hat{I}_{\gamma\beta)} + e_{\beta)}A_\gamma). \tag{2.56}$$

The correction (2.9) will appear only in second order. The frame itself is given by

$$\theta^\alpha = d(x^\alpha - \epsilon J_0^{\alpha\gamma}A_\gamma) - \epsilon J_0^{\alpha\gamma}\hat{I}_{\gamma\beta}dx^\beta. \tag{2.57}$$

We therefore find the following expressions

$$d\theta^\alpha = -\epsilon J_0^{\alpha\gamma}e_\delta\hat{I}_{\gamma\beta}dx^\delta dx^\beta = \tfrac{1}{2}\epsilon J_0^{\alpha\delta}e_\delta\hat{I}_{\beta\gamma}dx^\gamma dx^\beta, \tag{2.58}$$

$${C^\alpha}_{\beta\gamma} = \epsilon J_0^{\alpha\delta}e_\delta\hat{I}_{\beta\gamma}, \tag{2.59}$$

$$\omega_{\alpha\beta\gamma} = \tfrac{1}{2}\epsilon({J_{0[\alpha}}^\delta e_\delta\hat{I}_{\beta\gamma]} + {J_{0\beta}}^\delta e_\delta\hat{I}_{\alpha\gamma}). \tag{2.60}$$

The torsion obviously vanishes.

From (2.60) for the linearized Riemann tensor we obtain, using the cocycle condition, the expression

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\epsilon e^\eta\left(J_{0\eta[\gamma}e_{\delta]}\hat{I}_{\alpha\beta} + J_{0\eta[\alpha}e_{\beta]}\hat{I}_{\gamma\delta}\right). \tag{2.61}$$

For the Ricci curvature we find

$$R_{\beta\gamma} = -\tfrac{1}{2}\epsilon e^\zeta\left(J_{0\zeta(\beta}e^\alpha\hat{I}_{\gamma)\alpha} + J_0^\alpha{}_\zeta e_{(\beta}\hat{I}_{\gamma)\alpha}\right). \tag{2.62}$$

One more contraction yields the expression

$$R = -2\epsilon {J_0}^{\zeta\alpha}e_\zeta e^\beta\hat{I}_{\alpha\beta} \tag{2.63}$$

for the Ricci scalar. Using again the cocycle condition permits us to write this in the form

$$R = \epsilon\Delta\chi, \tag{2.64}$$

where the scalar trace component is defined as

$$\chi = J_0^{\alpha\beta}\hat{I}_{\alpha\beta}. \tag{2.65}$$

The Ricci scalar is a divergence. Classically it vanishes when the field equations are satisfied.

2.5 Coordinate invariance

In principle, because we use the frame formalism, covariance is assured at least to the semi-classical approximation. (We are not sure exactly what this would mean in general.) However since we base our construction on the commutator of two generators (coordinates) it is of interest to show this explicitly. The frame components $J_1^{\alpha\beta}$ of the perturbation of the commutator are not equal to the perturbation $I^{\alpha\beta}$ of the frame components but the two are related in a simple way. We have to first order the relation

$$J^{\alpha\beta} = \theta^\alpha_\mu \theta^\beta_\nu J^{\mu\nu} \tag{2.66}$$

from which we conclude that

$$\begin{aligned} J_1^{\alpha\beta} &= I^{\alpha\beta} - J_0^{[\alpha\gamma} \Lambda^{\beta]}_\gamma = I^{\alpha\beta} - J_0^{[\alpha\gamma} J_0^{\beta]\delta} \hat{\Lambda}_{\delta\gamma} \\ &= -(I^{\alpha\beta} + J_0^{\alpha\gamma} J_0^{\beta\delta} e_{[\gamma} A_{\delta]}) \\ &= -J_0^{\alpha\gamma} J_0^{\beta\delta} e_{[\gamma} (C + A)_{\delta]}. \end{aligned} \tag{2.67}$$

The second line is obtained using the solution (2.54) and the last uses the expression (2.52).

If we consider a first-order coordinate transformation of the form

$$x'^\mu = x^\mu + \epsilon B^\mu \tag{2.68}$$

we conclude that under this transformation the components of A transform as

$$A'_\alpha = A_\alpha + J^{-1}_{0\alpha\beta} B^\beta. \tag{2.69}$$

From the last line of the sequence of identities (2.67) we see that we can choose the coordinates to set $J'_1 = 0$. This is the Darboux theorem. We can also choose coordinates to set $A' = 0$. We cannot however set $I = 0$, fortunately since this would entail that the curvature vanish.

2.6 Covariant coordinates

There is a special case of particular interest, that in which the mixed components $J_0^{\mu\alpha}$ are constant and the matrix they form is invertible. We write then

$$x^\mu = J_0^{\mu\alpha} D_\alpha, \qquad D_\alpha = p_\alpha + \mathbf{A}_\alpha. \tag{2.70}$$

The interest in this decomposition resides in the properties of the 1-forms $\mathbf{A} = \mathbf{A}_\alpha \theta^\alpha$ and $\theta = -p_\alpha \theta^\alpha$ considered as gauge potentials. Let $Ų \subset Ą$ be the group of unitary elements of the algebra and define for arbitrary $\mathbf{A}$ and $g \in Ų$

$$\mathbf{A}' = g^{-1} \mathbf{A} g + g^{-1} dg. \tag{2.71}$$

Since

$$dg = e_\alpha g \theta^\alpha = -[\theta, g] \tag{2.72}$$

in the particular case with $\mathbf{A} = \theta$ we have $\theta' = \theta$. We conclude that, being the difference between two gauge potentials, the generators x^μ transform as adjoint representations of $\mathcal{U}$:

$$x'^\mu = g^{-1} x^\mu g. \tag{2.73}$$

This decomposition was introduced [12] in the particular case of a matrix algebra to describe the shift caused by spontaneous symmetry breaking and the generators were called [13, 14] covariant because of their transformation properties.

From (2.70) we deduce that

$$J^{\mu\nu} = J_0^{\mu\alpha} J_0^{\nu\beta} F_{\alpha\beta} \tag{2.74}$$

with

$$F = \frac{1}{2} F_{\alpha\beta}\theta^\alpha\theta^\beta = d\mathbf{A} + \mathbf{A}^2. \tag{2.75}$$

It would seem natural in this case at least to identify [11] noncommutativity as noncommutative electromagnetism and consider the action

$$S = \frac{1}{2} \operatorname{Tr} J_{\mu\nu} J^{\mu\nu} \tag{2.76}$$

as the action for commutator. We recall that noncommutative electromagnetism can contain Yang-Mills components with gauge group U_n for arbitrary n. It suffices that the algebra $\mathcal{A}$ contain a factor M_n.

3 The WKB approximation

In the commutative case the WKB dispersion relations followed from the field equations. In order to introduce the WKB approximation in noncommutative case, we suppose that the algebra $\mathcal{A}$ is a tensor product

$$\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_\omega \tag{3.1}$$

of a 'slowly-varying' factor $\mathcal{A}_0$ in which all amplitudes lie and a 'rapidly-varying' phase factor which is of order-of-magnitude ϵ so that only functions linear in this factor can appear. By 'slowly-varying' we mean an element f with a classical limit $\tilde{f}$ such that $\partial_\alpha \tilde{f} \lesssim \mu \tilde{f}$. The generic element f of the algebra is of the form then

$$f = f_0 + \epsilon \bar{f} e^{i\omega\phi} \tag{3.2}$$

where f_0 and $\bar{f}$ belong to $\mathcal{A}_0$. Because of the condition on ϵ the factor order does not matter and these elements form an algebra. The frequency parameter ω is so chosen that for an element f of $\mathcal{A}_0$ the estimate

$$[\phi, f] \simeq \bar{k}\mu \tag{3.3}$$

holds. The commutator $[f, e^{i\omega\phi}]$ is thus of order of magnitude

$$[f, e^{i\omega\phi}] \simeq \bar{k}\mu\omega. \tag{3.4}$$

The wave vector

$$\xi_\alpha = e_\alpha \phi \tag{3.5}$$

is normal to the surfaces of constant phase. We shall require also that the energy of the wave be such that it contribute not as source to the background field. This inequality can be written as

$$\epsilon\omega^2 \ll \mu^2. \tag{3.6}$$

It assures us also that to the approximation we are considering we need not pay attention to the order of the factors in the perturbation. We have in fact partially solved the system of equations without further approximation. One purpose of the following analysis is to verify that all constraints have been satisfied. We first recall the results one obtains in the classical case.

3.1 The commutative case

Classically, the vacuum equations are given by the condition that the Einstein tensor vanish. In the WKB approximation this yields in fact two equations. The leading order term, proportional to ω^2 is a dispersion relation ('E' for Einstein)

$$G_{\alpha\beta} = \tfrac{1}{2}\epsilon(i\omega)^2 \mathrm{Disp}_{E\alpha\beta} = 0 \tag{3.7}$$

with

$$\mathrm{Disp}_{E\alpha\beta} = -\xi^2\psi_{\alpha\beta} + \xi^\gamma\psi_{\gamma(\alpha}\xi_{\beta)} - \xi^\gamma\xi^\delta\psi_{\gamma\delta}\eta_{\alpha\beta}, \tag{3.8}$$

and

$$\psi_{\alpha\beta} = g_{1\alpha\beta} - \tfrac{1}{2}g^T\eta_{\alpha\beta}, \qquad g^T = g_{1\alpha}{}^\alpha. \tag{3.9}$$

If $\xi^2 = 0$ then it follows that

$$\xi^\gamma\psi_{\gamma\beta} = 0. \tag{3.10}$$

The second term in the expansion in frequency, the one proportional to ω, yields a conservation law

$$G_{\alpha\beta} = \tfrac{1}{2}\epsilon(i\omega)\mathrm{Cons}_{E\alpha\beta} = 0 \tag{3.11}$$

with

$$\mathrm{Cons}_{E\alpha\beta} = 2\xi^\gamma e_\gamma\psi_{\alpha\beta} + e_\gamma\xi^\gamma\psi_{\alpha\beta}. \tag{3.12}$$

This second equation can be interpreted [4, 5, 6] as a conservation of graviton number *in vacuo*. One easily sees that from it follows

$$\mathrm{Cons}_{E\beta\gamma}\psi^{\beta\gamma} = e_\alpha(\psi_{\beta\gamma}\psi^{\beta\gamma}\xi^\alpha) = 0. \tag{3.13}$$

With the Jacobi equations there is a similar doubling.

3.2 The quasi-commutative case

In the WKB approximation the perturbations Λ^α_β and $I^{\alpha\beta}$ are of the form

$$\Lambda^\alpha_\beta = \bar{\Lambda}^\alpha_\beta e^{i\omega\phi}, \qquad I^{\alpha\beta} = \bar{I}^{\alpha\beta} e^{i\omega\phi}, \tag{3.14}$$

where $\bar{\Lambda}^\alpha_\beta$ and $\bar{I}^{\alpha\beta}$ belong to $\mathcal{A}_0$. Therefore we have also

$$g_1^{\mu\nu} = \bar{g}^{\mu\nu} e^{i\omega\phi}. \tag{3.15}$$

Using $\xi_\alpha = e_\alpha \phi$ and $\eta^\alpha = J_0^{\alpha\beta}\xi_\beta$ we have

$$e_\alpha I_{\beta\gamma} = (i\omega\xi_\alpha \bar{I}_{\beta\gamma} + e_\alpha \bar{I}_{\beta\gamma}) e^{i\omega\phi}, \tag{3.16}$$

$$e_\alpha \Lambda_{\beta\gamma} = (i\omega\xi_\alpha \bar{\Lambda}_{\beta\gamma} + e_\alpha \bar{\Lambda}_{\beta\gamma}) e^{i\omega\phi}. \tag{3.17}$$

The cocycle condition replaces Einstein's equation to a certain extent. In the WKB approximation it becomes

$$\xi_\alpha \hat{I}_{\beta\gamma} + \xi_\beta \hat{I}_{\gamma\alpha} + \xi_\gamma \hat{I}_{\alpha\beta} = 0. \tag{3.18}$$

We multiply this equation by ξ^α and obtain

$$\xi^2 \hat{I}_{\beta\gamma} + \xi_{[\beta}\hat{I}_{\gamma]\alpha}\xi^\alpha = 0. \tag{3.19}$$

If $\xi^2 \neq 0$ then we conclude that

$$\hat{I}_{\beta\gamma} = -\xi^{-2}\xi_{[\beta}\hat{I}_{\gamma]\alpha}\xi^\alpha. \tag{3.20}$$

This is no restriction; it defines simply C_α by

$$i\omega C_\alpha = -\xi^{-2}\hat{I}_{\alpha\beta}\xi^\beta. \tag{3.21}$$

If $\xi^2 = 0$ then we conclude that

$$\xi_{[\beta}\hat{I}_{\gamma]\alpha}\xi^\alpha = 0. \tag{3.22}$$

This is a small restriction; the ξ_α must be a Petrov vector of $\hat{I}$. We shall improve on this in a particular case in Section 4. In terms of the scalar χ we obtain the relation

$$\hat{I}_{\alpha\beta}\eta^\beta = -\tfrac{1}{2}\chi\xi_\alpha. \tag{3.23}$$

Using the definition of η we find in the WKB approximation to first order

$$\omega_{\alpha\beta\gamma} = \tfrac{1}{2} i\omega\epsilon \left(\eta_{[\alpha}\hat{I}_{\beta\gamma]} + \eta_\beta \hat{I}_{\alpha\gamma}\right), \tag{3.24}$$

$$R_{\alpha\beta\gamma\delta} = -\tfrac{1}{2}\epsilon(i\omega)^2 \left(\eta_{[\gamma}\xi_{\delta]}\hat{I}_{\alpha\beta} - \eta_{[\alpha}\xi_{\beta]}\hat{I}_{\gamma\delta}\right), \tag{3.25}$$

$$R_{\beta\gamma} = -\tfrac{1}{2}\epsilon(i\omega)^2 \left(\xi_{(\beta}\eta^\alpha - \xi^\alpha \eta_{(\beta}\right) \hat{I}_{\gamma)\alpha}, \tag{3.26}$$

$$R = \epsilon(i\omega)^2 \chi \xi^2. \tag{3.27}$$

In average, the linear-order expressions vanish. We can calculate to second order if we average over several wavelengths. We use the approximations

$$\langle \hat{I}^{\alpha\beta} \rangle = 0, \qquad \langle \hat{I}^{\alpha\beta} \hat{I}^{\gamma\delta} \rangle = \frac{1}{2} \hat{\hat{I}}^{\alpha\beta} \hat{\hat{I}}^{\gamma\delta}. \tag{3.28}$$

Also as $e_\delta J^{-1}_{\beta\gamma} = -J^{-1}_{\beta\eta} e_\delta J^{\eta\zeta} J^{-1}_{\zeta\gamma}$ we can write $e_\delta J^{-1}_{\beta\gamma} = \epsilon e_\delta \hat{I}_{\beta\gamma}$. Therefore we find expanding (2.33) to second order the expression

$$\langle R_{\beta\gamma} \rangle = \tfrac{1}{2}\epsilon^2 (i\omega)^2 \Big(\bar{\chi} \xi^\alpha \eta_{(\gamma} \hat{\hat{I}}_{\beta)\alpha} + \tfrac{3}{4} \bar{\chi}^2 \xi_\beta \xi_\gamma + \eta^2 \hat{\hat{I}}_{\eta\beta} \hat{\hat{I}}^\eta{}_\gamma - \tfrac{1}{2} \eta_\beta \eta_\gamma \hat{\hat{I}}_{\alpha\eta} \hat{\hat{I}}^{\alpha\eta} \Big) \tag{3.29}$$

for the Ricci tensor and the expression

$$\langle R \rangle = \tfrac{1}{8} \epsilon^2 (i\omega)^2 (2\eta^2 \hat{\hat{I}}_{\alpha\beta} \hat{\hat{I}}^{\alpha\beta} + 7 \bar{\chi}^2 \xi^2) \tag{3.30}$$

for the Ricci scalar. We shall return to these formulae in Section 4.

3.3 The noncommutative lattice

As a lattice, the background noncommutativity is of considerable complexity, the contrary of a simple cubic lattice. It is in general non-periodic but in the WKB approximation we can assume periodicity since at the scale of the frequency J_0 is a constant 4×4 matrix. It is difficult to obtain general expressions for the modes and their dispersion relations; however, it is interesting to analyse them in more detail by considering a specific example. We take an arbitrary perturbation $\hat{I}_{\alpha\beta}$ with the wave vector ξ_α normalized so that $\xi_0 = -1$,

$$\hat{\hat{I}}_{\alpha\beta} = \begin{pmatrix} 0 & b_3 & -b_2 & e_1 \\ -b_3 & 0 & b_1 & e_2 \\ b_2 & -b_1 & 0 & e_3 \\ -e_1 & -e_2 & -e_3 & 0 \end{pmatrix}. \tag{3.31}$$

One easily sees that the cocyle condition is equivalent to the constraint $\vec{b} = -\vec{\xi} \times \vec{e}$ which is the part of the field equations for a plane wave, the Bianchi equations. Suppose that ξ is null and oriented along the z-axis, $\xi_\alpha = (0, 0, 1, -1)$. If $\hat{I}$ satisfies the cocycle condition we have

$$\hat{\hat{I}}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & -e_1 & e_1 \\ 0 & 0 & -e_2 & e_2 \\ e_1 & e_2 & 0 & e_3 \\ -e_1 & -e_2 & -e_3 & 0 \end{pmatrix}. \tag{3.32}$$

The perturbation $\hat{I}$ is of Petrov-type N if $\vec{\xi} \cdot \vec{e} = e_3 = 0$; this would be the second half of the Maxwell field equations. In this case, for arbitrary background noncom-

mutativity given by

$$J_{0\alpha\beta} = \begin{pmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix} \tag{3.33}$$

we can write the amplitude of the metric perturbation in the form

$$\bar{g}_{1\alpha\beta} = -J_{0(\alpha}{}^{\gamma}\hat{\hat{I}}_{\gamma\beta)} = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix}. \tag{3.34}$$

It is easy to check that by a change of coordinates we can set $P_{12} = 0$, $P_{22} = 0$. Introducing $e_1 = a\cos\gamma$, $e_2 = a\sin\gamma$, $B_2 + E_1 = A\sin\Gamma$, $B_1 - E_2 = A\cos\Gamma$ the remaining part P_{11} can be decomposed

$$P_{11} = aA \begin{pmatrix} \sin(\gamma+\Gamma) & \cos(\gamma+\Gamma) \\ \cos(\gamma+\Gamma) & -\sin(\gamma+\Gamma) \end{pmatrix} + aA \begin{pmatrix} \sin(\gamma-\Gamma) & 0 \\ 0 & \sin(\gamma-\Gamma) \end{pmatrix} \tag{3.35}$$

into a trace-free part and a trace. The gravitational wave is polarized, and though the polarization is fixed in terms of $\gamma+\Gamma$, it can be arbitrary. In addition there is a scalar wave, the trace. In the case when $e_3 \neq 0$ the perturbation $\hat{I}$ is not of Petrov type N; the additional gravitational mode is a constraint mode.

4 The Poisson Energy and conservation laws

We have associated a gravitational field to the noncommutative structure with the map (1.7). We would like to consider now this structure as an effective field and estimate its energy-momentum ('Poisson energy'). We are confronted immediately with the choice of the position of the extra term in the Einstein equations. We can place it on the right-hand side and consider it on the same level as any matter source. We can also keep it on the left-hand side and consider it as a noncommutative modification of the curvature. First however we make some preliminary remarks about conservation laws.

From (2.62-2.63) for the Einstein tensor we obtain

$$G_{\beta\gamma} = -\tfrac{1}{2}\epsilon\Big(J_{0\zeta(\beta}e^{\zeta}e^{\alpha}\hat{I}_{\gamma)\alpha} + J_0{}^{\alpha\zeta}e_{\zeta}e_{(\beta}\hat{I}_{\gamma)\alpha} - 2\eta_{\beta\gamma}J_{0\zeta\delta}e^{\zeta}e_{\alpha}\hat{I}^{\delta\alpha}\Big). \tag{4.1}$$

In general the Einstein tensor does not vanish. A conservation equation of the associated energy-momentum tensor in linear approximation is easy to verify. Applying the cocycle condition and keeping in mind that, to linear order in ϵ, $e_\alpha e_\beta = e_\beta e_\alpha$, we obtain

$$\begin{aligned} e^{\beta}G_{\beta\gamma} &= -\tfrac{1}{2}\epsilon\Big(J_0{}^{\alpha\zeta}e^{\beta}e_{\zeta}e_{\gamma}\hat{I}_{\beta\alpha} + J_0{}^{\alpha\zeta}e^{\beta}e_{\zeta}e_{\beta}\hat{I}_{\gamma\alpha} - 2J_{0\zeta\delta}e_{\gamma}e^{\zeta}e_{\alpha}\hat{I}^{\delta\alpha}\Big) \\ &= -\tfrac{1}{2}\epsilon J_0{}^{\delta\zeta}e_{\zeta}e^{\alpha}(e_{\alpha}\hat{I}_{\gamma\delta} - e_{\gamma}\hat{I}_{\alpha\delta}) \\ &= \tfrac{1}{2}\epsilon J_{0\delta\zeta}e^{\zeta}e^{\alpha}e^{\delta}\hat{I}_{\alpha\gamma} = 0. \end{aligned} \tag{4.2}$$

As we shall see, the conservation law holds in an important special case in quadratic order too.

4.1 Canonical orientation

To the extent that the noncommutative background is analogous to a lattice, the perturbations can be considered as elastic vibrations or phonons. This analogy however is tenuous at the approximation we are considering since we have excluded any resonance phenomena. These could appear if we allowed larger-amplitude waves with energy sufficient to change the background. The case we shall now focus to would then be analogous to a phonon propagating along one of the axes of a regular cubic lattice. In the special case in which it is also Petrov vector of the perturbation the dispersion relations become clearer.

Assume then that η and ξ are parallel and set

$$\eta^\alpha = J_0^{\alpha\beta}\xi_\beta = \lambda\xi^\alpha. \tag{4.3}$$

It follows from (3.23) that the vector ξ is an eigenvector of J also to second order. Equation (3.25) yields for the Riemann curvature tensor to linear order

$$R_{\alpha\beta\gamma\delta} = 0. \tag{4.4}$$

The dispersion relation

$$\xi^2 = 0 \tag{4.5}$$

follows from (3.19).

We stress that we have no action and that this dispersion relation was not obtained from field equations. It is valid however only in the case of wave propagation satisfying the relation (4.3). There is a certain amount of obscurity surrounding the role of the cocycle condition, to what extent and how it can be used to replace the field equations. Suppose for simplicity that the scalar field χ vanishes. Using the dual object $\hat{I}^*$ one can write the cocycle condition as the condition

$$\hat{I}^{*\alpha\beta}\xi_\beta = 0. \tag{4.6}$$

On the other hand Equation (3.23) in the particular orientation we have chosen for the wave propagation is written as

$$\hat{I}_{\alpha\beta}\xi^\beta = 0. \tag{4.7}$$

We can conclude then that for any complex c the difference

$$H = \hat{I}^* - c\hat{I} \tag{4.8}$$

is orthogonal to the propagation vector. We are considering the WKB approximation which implies that the essential direction is in fact ξ. We can conclude then that 'essentially' we have $H = 0$. But this is a modified self-duality condition, that is a condition which equates a dynamical object with a topological one.

In quadratic order, using the dispersion relation, we find that the expression (3.29) simplifies to

$$\langle R_{\beta\gamma}\rangle = -\tfrac{1}{8}\epsilon^2(i\omega)^2(\bar{\chi}^2 + 2\lambda^2 \hat{\hat{I}}_{\alpha\eta}\hat{\hat{I}}^{\alpha\eta})\xi_\beta\xi_\gamma \tag{4.9}$$

for the Ricci tensor. The corresponding expression for the Ricci scalar vanishes and we obtain for the Einstein tensor the average value

$$\langle G_{\beta\gamma}\rangle = -\rho\xi_\beta\xi_\gamma \tag{4.10}$$

with

$$\rho = -\tfrac{1}{8}(\epsilon\omega)^2(\bar{\chi}^2 + 2\lambda^2 \hat{\hat{I}}_{\alpha\eta}\hat{\hat{I}}^{\alpha\eta}). \tag{4.11}$$

The energy-momentum is that of a null dust with a density ρ.

In the WKB approximation we can, just as in the classical case, derive a conservation law for ρ which has a natural interpretation as graviton-number conservation. The conservation law however now can be derived directly from the cocycle condition. If we multiply the cocycle condition (2.51) by ξ^α we obtain

$$\xi^\alpha e_\alpha \hat{I}_{\beta\gamma} + \xi^\alpha e_\beta \hat{I}_{\gamma\alpha} + \xi^\alpha e_\gamma \hat{I}_{\alpha\beta} = 0. \tag{4.12}$$

We also have

$$e^\alpha(\xi_\alpha \hat{I}_{\beta\gamma} + \xi_\beta \hat{I}_{\gamma\alpha} + \xi_\gamma \hat{I}_{\alpha\beta}) = 0. \tag{4.13}$$

Adding these two equations, using (3.23), (4.3) and not forgetting that $e_\alpha\xi_\beta = e_\beta\xi_\alpha$ in our approximation, we find

$$\begin{aligned}&(\xi^\alpha e_\alpha \hat{I}_{\beta\gamma} + e^\alpha(\xi_\alpha \hat{I}_{\beta\gamma}))\hat{I}^{\beta\gamma} + 2(\xi^\alpha e_\beta \hat{I}_{\gamma\alpha} + e^\alpha(\xi_\beta \hat{I}_{\gamma\alpha}))\hat{I}^{\beta\gamma}\\ &= e_\alpha(\xi^\alpha \hat{I}^{\beta\gamma}\hat{I}_{\beta\gamma}) + 2e^\alpha(\xi^\beta \hat{I}^{\gamma\alpha}\hat{I}_{\beta\gamma}) = 0.\end{aligned} \tag{4.14}$$

The conservation law

$$e^\alpha(\rho\xi_\alpha) = 0 \tag{4.15}$$

follows and from it the conservation of the effective source,

$$e^\alpha(\rho\xi_\alpha\xi_\beta) = 0. \tag{4.16}$$

To interpret the additional term we have isolated as the energy-momentum of an external field,

$$G_{\beta\gamma} = -16\pi G_N T_{\beta\gamma}, \tag{4.17}$$

the sign of ρ should be non-negative. However, as

$$\rho = -\tfrac{1}{8}(\epsilon\omega)^2(\bar{\chi}^2 + 2\lambda^2 \hat{\hat{I}}_{\alpha\eta}\hat{\hat{I}}^{\alpha\eta}) = \tfrac{1}{4}(\epsilon\omega\lambda\vec{e})^2 - \tfrac{1}{4}(\epsilon\omega\lambda\vec{b})^2 - \tfrac{1}{8}(\epsilon\omega\bar{\chi})^2, \tag{4.18}$$

the matter density does not have a fixed sign, unless of course one place restrictions on the relative importance of the space-time and space-space commutation relations. This exactly is one of the properties which can explain the acceleration of the universe [16] and it makes the 'Poisson energy' a possible candidate for dark energy. We shall examine this possibility in a subsequent article.

4.2 Sparling's forms

In the Cartan frame formalism the field equations are most elegantly (and, once one is familiar with it, easily) written as the vanishing of a 3-form. This follows from the identity

$$G_{\alpha\beta} * \theta^{\beta} = -\tfrac{1}{2}\Omega^{\beta\gamma} * \theta_{\alpha\beta\gamma}. \tag{4.19}$$

We have here introduced

$$* \theta^{\alpha} = \frac{1}{3!}\epsilon^{\alpha\beta\gamma\delta}\theta_{\beta}\theta_{\gamma}\theta_{\delta} \tag{4.20}$$

as well as its inverse

$$* \theta^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma\delta}\theta_{\delta}. \tag{4.21}$$

Since the signature is that of Minkowski we find that $*^2 = -1$.

We write the energy-momentum of the gravitational field in terms of a vector-valued 3-form ('S' for Sparling)

$$\tau_{S\alpha} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}(\delta^{\beta}_{\zeta}\omega^{\gamma\eta}\omega_{\eta}{}^{\delta} - \omega^{\gamma\delta}\omega^{\beta}{}_{\zeta})\theta^{\zeta} \tag{4.22}$$

which has the property [15] that it is exact if and only if the Einstein field equations are satisfied. If this be so one sees that the total energy-momentum is given as the integral over the sphere at infinity of the Sparling 2-form

$$\sigma_{\alpha} = -\tfrac{1}{2}\omega^{*}_{\alpha\beta}\theta^{\beta}, \qquad \omega^{*}_{\alpha\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\omega^{\gamma\delta}. \tag{4.23}$$

Since in general there is a preferred frame, that canonically aligned with respect to the eigenvectors of the conformal tensor, one can claim that the 2-form itself and not only the integral thereof is well-defined.

We can then also consider the extra terms we obtained in the previous section to be due to a noncommutative extension of the curvature and write Equation (4.10) as a vacuum equation

$$\langle G_{\beta\gamma}\rangle + \rho\xi_{\beta}\xi_{\gamma} = 0. \tag{4.24}$$

The field equations (4.10) can then be written as the condition ('P' for Poisson)

$$G_{\alpha\beta} * \theta^{\beta} - \tau_{P\alpha} = 0. \tag{4.25}$$

Classically one has the identity

$$G_{\alpha\beta} * \theta^{\beta} + \tau_{S\alpha} = d\sigma_{\alpha}. \tag{4.26}$$

We can write then

$$\tau_{P\alpha} + \tau_{S\alpha} = d\sigma_{\alpha}. \tag{4.27}$$

The vacuum field equations (4.24) are the integrability conditions for the modified system

$$d\tau_{\alpha} = 0, \qquad \tau_{\alpha} = \tau_{P\alpha} + \tau_{S\alpha}. \tag{4.28}$$

However, only when we have calculated the modification in a few more examples can we hope to lift the change of the 3-form to a change of the curvature form.

5 Conclusions

The formalism on which the article has been based is one with a preferred frame. It is in a sense then gauge-fixed from the beginning. We have shown that the degrees-of-freedom or basic modes of the resulting theory of gravity can be put in correspondence with those of the noncommutative structure. As an application of the formalism we have considered a high-frequency perturbation of the metric. In the classical theory it follows from the field equations that the perturbation must satisfy a dispersion relation and a conservation law. We show that these remain valid in the noncommutative extension of the frame formalism and that they are consequences of a cocycle condition on the corresponding perturbation of the Poisson structure.

We have also shown that the perturbation of the Poisson structure contributes to the energy-momentum as an additional effective source of the gravitational field. Although the explicit form of this contribution, the Poisson energy, was calculated only in a linearized, high-frequency approximation it is certainly significant in a more general context. We stress that because of the identification of the gravitational field with the Poisson structure the perturbation of the latter is in fact a reinterpretation of a perturbation of the former and not an extra field. The difference with classical gravity lies in the choice of field equations. In the WKB approximation this amounts only to a modification of the conserved quantity.

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